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THE CONVERTIBILITY OF $\operatorname{Ext}_R^n(-, A)$

BY

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ABSTRACT. Let R be a commutative ring and $\operatorname{Mod}(R)$ the category of R-modules. Call a contravariant functor F: $\operatorname{Mod}(R) \longrightarrow \operatorname{Mod}(R)$ convertible if for every direct system $\{X_\alpha\}$ in $\operatorname{Mod}(R)$ there is a natural isomorphism γ: $F(\varinjlim X_\alpha) \longrightarrow \varprojlim F(X_\alpha)$. If A is in $\operatorname{Mod}(R)$ and n is a positive integer then $\operatorname{Ext}_R^n(-, A)$ is not in general convertible. The purpose of this paper is to study the convertibility of Ext, and in so doing to find out more about Ext as well as the modules A that make $\operatorname{Ext}_R^n(-, A)$ convertible for all n.

It is shown that $\operatorname{Ext}_R^n(-, A)$ is convertible for all A having finite length and all n. If R is Noetherian then A can be Artinian, and if R is semilocal Noetherian then A can be linearly compact in the discrete topology. Characterizations are studied and it is shown that if A is a finitely generated module over the semilocal Noetherian ring R, then $\operatorname{Ext}_R^1(-, A)$ is convertible if and only if A is complete in the J-adic topology where J is the Jacobson radical of R. Morita-duality is characterized by the convertibility of $\operatorname{Ext}_R^1(-, R)$ when R is a Noetherian ring, a reflexive ring or an almost maximal valuation ring. Applications to the vanishing of Ext are studied.

Introduction. Let D be a category with direct limits and D' a category with inverse limits. Call a contravariant functor F: $D \longrightarrow D'$ *convertible* if for every direct system $\{X_\alpha\}$ in D there is a natural isomorphism γ: $F(\varinjlim X_\alpha) \longrightarrow \varprojlim F(X_\alpha)$. If R is a ring we let $\operatorname{Mod}(R)$ be the category of right R-modules and $\operatorname{Mod}(Z)$ the category of abelian groups. If G: $\operatorname{Mod}(R) \longrightarrow \operatorname{Mod}(Z)$ is a contravariant functor and $\{X_\alpha\}$ is a direct system in $\operatorname{Mod}(R)$, then there is a natural group homomorphism σ: $G(\varinjlim X_\alpha) \longrightarrow \varprojlim G(X_\alpha)$ defined by $\sigma(x) = (G(g_\alpha)(x))$ for $x \in G(\varinjlim X_\alpha)$ where the maps $\{g_\alpha\}$ are those corresponding to $\varinjlim X_\alpha$. Thus G is convertible if σ is an isomorphism for all direct systems in $\operatorname{Mod}(R)$. For any module A in $\operatorname{Mod}(R)$ it is well known that $\operatorname{Hom}_R(-, A)$ is convertible. However, when Hom is replaced by Ext^n for a positive integer n, then $\operatorname{Ext}_R^n(-, A)$ is not in general convertible.

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The purpose of this paper is to study the convertibility of Ext, and in so doing to find out more about Ext as well as the modules A that make $\mathrm{Ext}_R^n(-, A)$ convertible for all n. If R is commutative we let the domain and range categories be the category of R-modules since σ is then an R-homomorphism.

Let R and S be rings, B an R-S bimodule, C an injective right S-module and $A = \mathrm{Hom}_S(B, C)$. Then it is shown that $\mathrm{Ext}_R^n(-, A)$ is convertible for all n. This leads us to the study of U-reflexive modules where U is an injective cogenerator. In this regard we are able to show that if R is a commutative ring then $\mathrm{Ext}_R^n(-, A)$ is convertible for all modules A having finite length and all n. Further, if R is Noetherian it follows that A can be Artinian and if R is semilocal Noetherian it follows that A can be linearly compact in the discrete topology.

Next we study characterizations of a module via the convertibility of Ext. It is shown that if R is a commutative semilocal Noetherian ring and A is a finitely generated R-module then $\mathrm{Ext}_R^1(-, A)$ is convertible if and only if A is complete in the J-adic topology where J is the Jacobson radical of R. Thus $\mathrm{Ext}_R^1(-, A)$ becomes a "completion" functor. We take the case $A = R$ and show that if R is a commutative Noetherian ring, a reflexive ring or an almost maximal valuation ring, then $\mathrm{Ext}_R^1(-, R)$ is convertible if and only if R has a Morita-duality.

In the last section we include applications to the vanishing of Ext along with some remarks about the usefulness, in studying the convertibility of Ext, of a spectral sequence of Roos [15] together with the theory of the right derived functors of inverse limit given by Jensen [6].

1. Preliminaries and tools. Throughout this paper all rings will have an identity and all modules will be unitary. All modules over a ring R will be understood to be right R-modules unless specifically stated otherwise. All notation and terminology involving homological algebra will be standard and can be found in the standard work [3]. When we say that $\{X_\alpha\}$ is a direct system or an inverse system we shall always mean that the index set is a partially ordered directed set. We will not indicate the index set and the maps corresponding to $\{X_\alpha\}$ unless they are needed. If R is a ring and A is an R-module then the injective envelope of A is denoted by $E(A)$. An R-module U is called a *cogenerator* (in the category of R-modules) if it contains a copy of the injective envelope of every simple R-module. U is called a *minimal injective cogenerator* if it is isomorphic to $E(\bigoplus_M R/M)$ where M ranges over all the maximal ideals of R.

If A and U are right (left) R-modules and $S = \mathrm{Hom}_R(U, U)$, then U and $\mathrm{Hom}_R(A, U)$ are naturally left (right) S-modules by agreeing to write the elements of S on the left (right) of their arguments. Therefore $\mathrm{Hom}_S(\mathrm{Hom}_R(A, U), U)$ is a right (left) R-module and there is a natural R-homomorphism

$$\phi_1: A \to \mathrm{Hom}_S(\mathrm{Hom}_R(A, U), U)$$

defined by $\phi_1(a)(f) = f(a)$ for all $a \in A$ and $f \in \mathrm{Hom}_R(A, U)$. If ϕ_1 is a monomorphism A is called *U-torsionless* and if ϕ_1 is an isomorphism A is called *U-reflexive.* In the case where R is a commutative ring there is a natural R-homomorphism

$$\phi_2: A \to \mathrm{Hom}_R(\mathrm{Hom}_R(A, U), U)$$

defined the same as ϕ_1. In this case when we refer to the concepts of torsionless or reflexive we will mean that ϕ_2 is a monomorphism or an isomorphism, unless we specifically state otherwise. It is easy to see that A is U-torsionless if and only if for every nonzero $a \in A$ there exists an $f \in \mathrm{Hom}_R(A, U)$ such that $f(a) \neq 0$.

We now state three well-known equivalent conditions for an R-module U to be a cogenerator:

(a) U is a cogenerator.

(b) Every R-module is U-torsionless.

(c) Every R-module is contained in a product of copies of U.

The following proposition is the fundamental tool that we use to find modules that make Ext convertible.

Proposition 1.1. *Let R and S be rings and B an R-S bimodule with R acting on the left and S acting on the right. Let C be an injective right S-module and denote the right R-module $\mathrm{Hom}_S(B, C)$ by A. Then $\mathrm{Ext}_R^n(-, A)$ is convertible for all n.*

Proof. Let $\{X_\alpha\}$ be a direct system of R-modules. Since C is an injective right S-module it follows that

$$\begin{aligned}\mathrm{Ext}_R^n(\varinjlim X_\alpha, A) &= \mathrm{Ext}_R^n(\varinjlim X_\alpha, \mathrm{Hom}_S(B, C)) \cong \mathrm{Hom}_S(\mathrm{Tor}_n^R(\varinjlim X_\alpha, B), C)\\ &\cong \mathrm{Hom}_S(\varinjlim \mathrm{Tor}_n^R(X_\alpha, B), C) \cong \varprojlim \mathrm{Hom}_S(\mathrm{Tor}_n^R(X_\alpha, B), C)\\ &\cong \varprojlim \mathrm{Ext}_R^n(X_\alpha, \mathrm{Hom}_S(B, C)) = \varprojlim \mathrm{Ext}_R^n(X_\alpha, A).\end{aligned}$$

The isomorphisms follow because of [3, Chapter VI, Proposition 5.1] and because Tor commutes with direct limit and $\mathrm{Hom}_S(-, C)$ is convertible.

Corollary 1.2. *Let R be a commutative ring. Then there exists a ring extension S of R such that $\mathrm{Ext}_S^n(-, S)$ is convertible for all n.*

Proof. Let U be an injective cogenerator for R and set $S = \mathrm{Hom}_R(U, U)$. U is a left S-module in the usual way by defining $sx = s(x)$ for $s \in S$ and $x \in U$. So it follows from Proposition 1.1 (with R and S interchanged) that $\mathrm{Ext}_S^n(-, S)$ is convertible for all n. Since U is a cogenerator it follows that the R-homomorphism β: $R \to S$ defined by $\beta(r)(x) = rx$ for $r \in R$ and $x \in U$ is a ring monomorphism.

Remarks. (1) It is clear from the proof of Corollary 1.2 that there are many rings S containing R such that $\operatorname{Ext}_S^n(-, S)$ is convertible for all n. An unanswered question is the following: Is there a "minimal" ring S containing R such that $\operatorname{Ext}_S^n(-, S)$ is convertible for all n?

(2) Considering the proof of Corollary 1.2 we state a converse: If S is a ring such that $\operatorname{Ext}_S^n(-, S)$ is convertible for all n, then there is a ring R contained in the center of S and an injective R-module U such that $S = \operatorname{Hom}_R(U, U)$. We show later that this converse is true (in fact $R = S$) in the three cases where S is a commutative Noetherian ring, a reflexive ring or an almost maximal valuation ring. It is not known if the converse is true in general.

We now proceed to a duality theorem for reflexive modules which will be used later. We need some notation and a lemma, whose proof is standard and therefore omitted.

Notation. Let R be a commutative ring and let A and U be two R-modules. When there is no confusion about U we will write $A^* = \operatorname{Hom}_R(A, U)$ and $A^{**} = (A^*)^*$. If S is a subset of A we denote the *annihilator of* S *in* A^* by $\operatorname{Ann}_{A^*}(S) = \{f \in A^* \mid f(x) = 0 \text{ for all } x \in S\}$. If T is a subset of A^* we denote the *annihilator of* T *in* A by $\operatorname{Ann}_A(T) = \{a \in A \mid f(a) = 0 \text{ for all } f \in T\}$. If C is a submodule of A then it is easy to see that $\operatorname{Ann}_{A^*}(C) \cong \operatorname{Hom}_R(A/C, U)$ and $C \subset \operatorname{Ann}_A(\operatorname{Ann}_{A^*}(C))$. If U is a cogenerator we have the equality $C = \operatorname{Ann}_A(\operatorname{Ann}_{A^*}(C))$.

Lemma 1.3. *Let R be a commutative ring, U an injective cogenerator and $0 \to A \to B \to C \to 0$ an exact sequence of R-modules. Then B is U-reflexive if and only if A and C are U-reflexive.*

Proposition 1.4. *Let R be a commutative ring, U a cogenerator and A a U-reflexive R-module. Then*

(a) *There is a one to one order inverting correspondence between the submodules C of A and D of A^* given by $C \leftrightarrow \operatorname{Ann}_{A^*}(C)$ and $D \leftrightarrow \operatorname{Ann}_A(D)$ and we have the equalities $C = \operatorname{Ann}_A(\operatorname{Ann}_{A^*}(C))$ and $D = \operatorname{Ann}_{A^*}(\operatorname{Ann}_A(D))$.*

(b) *A is Noetherian (Artinian) if and only if A^* is Artinian (Noetherian).*

(c) *If U is injective then all submodules and factor modules (as well as their finite direct sums) of A and A^* are U-reflexive. In particular C and $A^*/\operatorname{Ann}_{A^*}(C)$ are U-duals of each other as are D and $A/\operatorname{Ann}_A(D)$ where C is a submodule of A and D is a submodule of A^*.*

Proof. (a) Since U is a cogenerator we have $C = \operatorname{Ann}_A(\operatorname{Ann}_{A^*}(C))$ as mentioned above. Let D be a submodule of A^*. Then by definition we have $D \subset \operatorname{Ann}_{A^*}(\operatorname{Ann}_A(D))$. To show the opposite inclusion let $f \in \operatorname{Ann}_{A^*}(\operatorname{Ann}_A(D))$ and suppose by way of contradiction that $f \notin D$. Then $f + D$ is a nonzero element of A^*/D and A^*/D is U-torsionless. Therefore there exists an element $F \in \mathrm{Hom}_R(A^*/D, U)$ such that $F(f + D) \neq 0$. But we have a natural isomorphism $\mathrm{Hom}_R(A^*/D, U) \cong \mathrm{Ann}_{A^{**}}(D)$ so that there exists $G \in \mathrm{Ann}_{A^{**}}(D)$ such that $G(f) \neq 0$. Since A is U-reflexive we have $\mathrm{Ann}_{A^{**}}(D) \cong \mathrm{Ann}_A(D)$. Let $\phi\colon A \to A^{**}$ be the natural isomorphism. Then there exists an element $a \in \mathrm{Ann}_A(D)$ such that $G = \phi(a)$. Therefore $f(a) = \phi(a)(f) = G(f) \neq 0$ contrary to the fact that f is in $\mathrm{Ann}_{A^*}(\mathrm{Ann}_A(D))$. So $D = \mathrm{Ann}_{A^*}(\mathrm{Ann}_A(D))$ and the one to one correspondence is now clear.

(b) Follows directly from part (a).

(c) If U is injective it follows from Lemma 1.3 that all the modules considered are U-reflexive. Consider the exact sequence $0 \to C \to A \to A/C \to 0$. By applying $\mathrm{Hom}_R(-, U)$ to this sequence we obtain $\mathrm{Hom}_R(C, U) \cong A^*/\mathrm{Ann}_{A^*}(C)$. Since $A \cong A^{**}$ we obtain in a similar manner the natural isomorphism $\mathrm{Hom}_R(D, U) \cong A/\mathrm{Ann}_A(D)$. On the other hand we have

$$\mathrm{Hom}_R(A^*/\mathrm{Ann}_{A^*}(C), U) \cong \mathrm{Ann}_{A^{**}}(\mathrm{Ann}_{A^*}(C)) \cong \mathrm{Ann}_A(\mathrm{Ann}_{A^*}(C)) = C$$

and

$$\mathrm{Hom}_R(A/\mathrm{Ann}_A(D), U) \cong \mathrm{Ann}_{A^*}(\mathrm{Ann}_A(D)) = D.$$

2. Modules that make Ext convertible.

Proposition 2.1. *Let R be a commutative ring, U an injective R-module and A a U-reflexive R-module. Then $\mathrm{Ext}_R^n(-, A)$ is convertible for all n.*

Proof. Follows from Proposition 1.1 by letting $S = R$, $C = U$ and $B = \mathrm{Hom}_R(A, U)$.

Proposition 2.2. *Let R be a commutative ring, U a minimal injective cogenerator and A an R-module of finite length. Then $\mathrm{Hom}_R(A, U)$ has finite length and its length is equal to that of A.*

Proof. If B is an R-module we will denote the length of B by $L(B)$. The proof will be by induction on length. So suppose $L(A) = 1$. Then there is a maximal ideal M of R such that $A \cong R/M$. The claim is that $R/M \cong \mathrm{Hom}_R(R/M, U)$. We have $\mathrm{Hom}_R(R/M, U) \cong \mathrm{Ann}_U(M)$ and we may assume that $U = E(\bigoplus_\alpha R/M_\alpha)$ where M_α ranges over all the maximal ideals of R. Therefore $R/M \subset \mathrm{Ann}_U(M)$. To show the opposite inclusion let $x \in \mathrm{Ann}_U(M)$, $x \neq 0$. Since $x \in U$ there exists an element $t \in R$ such that $tx \in \bigoplus_\alpha R/M_\alpha$ and $tx \neq 0$. Since $Mx = 0$ it follows that $t \notin M$. Therefore we have $R = M + Rt$ so that there exist elements $m \in M$ and $r \in R$ such that $1 = m + rt$. We note that $mx = 0$ so that $x = rtx$ which says that $x \in \bigoplus_\alpha R/M_\alpha$. Let $r_\alpha + M_\alpha$ be the αth component of x in $\bigoplus_\alpha R/M_\alpha$. Then $Mr_\alpha \subset M_\alpha$. So either $r_\alpha \in M_\alpha$ or $M = M_\alpha$. In other words we have $x \in R/M$. Hence $R/M = \mathrm{Ann}_U(M) \cong \mathrm{Hom}_R(R/M, U)$ so the proposition is true when $L(A) = 1$. Now suppose that $n > 1$ and the proposition is true for all R-modules having length less than n. Let $L(A) = n$. Then there exists an exact sequence $0 \to S \to A \to B \to 0$ where S is a simple R-module. Since length is an additive function we have $L(B) = n - 1$. We apply $\mathrm{Hom}_R(-, U)$ to the exact sequence and obtain another exact sequence $0 \to \mathrm{Hom}_R(B, U) \to \mathrm{Hom}_R(A, U) \to \mathrm{Hom}_R(S, U) \to 0$. The induction assumption applies to S and B so that $L(\mathrm{Hom}_R(B, U)) = L(B) = n - 1$ and $L(\mathrm{Hom}_R(S, U)) = L(S) = 1$. Therefore $L(\mathrm{Hom}_R(A, U)) = L(\mathrm{Hom}_R(B, U)) + L(\mathrm{Hom}_R(S, U)) = n - 1 + 1 = n = L(A)$.

Corollary 2.3. *Let R be a commutative ring and U a minimal injective cogenerator. Then every R-module of finite length is U-reflexive.*

Proof. Let A be an R-module of finite length. Since U is a cogenerator the following sequence is exact:

$$0 \to A \xrightarrow{\phi} \mathrm{Hom}_R(\mathrm{Hom}_R(A, U), U) \to \mathrm{Coker}\,\phi \to 0.$$

But $L(A) = L(\mathrm{Hom}_R(A, U)) = L(\mathrm{Hom}_R(\mathrm{Hom}_R(A, U), U))$ by Proposition 2.2. Therefore $L(\mathrm{Coker}\,\phi) = 0$ so that $\mathrm{Coker}\,\phi = 0$. Hence A is U-reflexive.

The next result is now clear because of Proposition 2.1.

Corollary 2.4. *Let R be a commutative ring and A an R-module of finite length. Then $\mathrm{Ext}^n_R(-, A)$ is convertible for all n.*

The next result is an extension of the Matlis-duality theorems [8, Theorem 4.2 and Corollary 4.3] to the semilocal case.

Proposition 2.5. *Let R be a commutative semilocal Noetherian ring which is complete in the J-adic topology where J is the Jacobson radical of R and let U be a minimal injective cogenerator. Then R is U-reflexive and $\mathrm{Hom}_R(-, U)$ establishes a category equivalence between the category of finitely generated R-modules and the category of Artinian R-modules.*

Proof. Let $M_1, \cdots, M_k$ be the maximal ideals of R so that $J = \bigcap_{i=1}^k M_i$ and $U = E(\bigoplus_{i=1}^k R/M_i)$. It is easy to see that if $i \neq j$ then $\mathrm{Hom}_R(E(R/M_i), E(R/M_j)) = 0$. It follows from [8, Theorem 3.7] that $\mathrm{Hom}_R(E(R/M_i), E(R/M_i)) \cong \hat{R}_{M_i}$, the completion of R_{M_i} in the $M_iR_{M_i}$-adic topology. Therefore we have

$$\mathrm{Hom}_R(U, U) \cong \bigoplus_{i=1}^{k} \mathrm{Hom}_R(E(R/M_i), E(R/M_i)) \cong \bigoplus_{i=1}^{k} \hat{R}_{M_i} = \bigoplus_{i=1}^{k} (\varprojlim R_{M_i}/M_i^n R_{M_i})$$
$$\cong \bigoplus_{i=1}^{k} (\varprojlim R/M_i^n) \cong \varprojlim \left(\bigoplus_{i=1}^{k} R/M_i^n \right) \cong \varprojlim R/J^n = R.$$

The isomorphisms are all natural so it follows that R is U-reflexive. Now let A be a finitely generated R-module generated by, say n, elements. Set $R^n = R \oplus \cdots \oplus R$ (n times). Then there is an exact sequence $R^n \to A \to 0$. We apply the functor $\mathrm{Hom}_R(-, U)$ to obtain the exact sequence $0 \to \mathrm{Hom}_R(A, U) \to U^n$. Since U is Artinian it follows that $\mathrm{Hom}_R(A, U)$ is Artinian. Similarly if A is an Artinian R-module, then it has a finitely generated socle so that there exists an integer n and an exact sequence $0 \to A \to U^n$ which leads to the exact sequence $R^n \cong \mathrm{Hom}_R(U^n, U) \to \mathrm{Hom}_R(A, U) \to 0$. Therefore $\mathrm{Hom}_R(A, U)$ is finitely generated and the result then follows from Proposition 1.4 because A is U-reflexive for A in either category.

Notation. Let R be a commutative Noetherian ring, A an R-module and M a maximal ideal of R. The *M-primary component of* A is the submodule $X_M(A) = \{x \in A \mid M^k x = 0 \text{ for some } k > 0\}$. A is called *M-primary* if $A = X_M(A)$. We say that M *belongs to* A if $X_M(A) \neq 0$. If $\{M_\alpha\}$ is a set of maximal ideals of R we say that A *belongs to* $\{M_\alpha\}$ if there is at least one $M_\beta \in \{M_\alpha\}$ such that $X_{M_\beta}(A) \neq 0$ and $X_M(A) = 0$ for all $M \notin \{M_\alpha\}$. If A is M-primary then there are natural R-isomorphisms $A \cong A \otimes_R R_M \cong A \otimes_R \hat{R}_M$ making A into an R_M-module as well as an $\hat{R}_M$-module [10, Proposition 2]. If A is an Artinian R-module then there are only a finite number of maximal ideals $M_1, \cdots, M_k$ belonging to A and $A = X_{M_1}(A) \oplus \cdots \oplus X_{M_k}(A)$ [10, Theorem 1]. It also follows in this case that $A_{M_i} \cong X_{M_i}(A)$ for each $i = 1, \cdots, k$. If A is an M-primary R-module and U is a minimal injective cogenerator then it is easy to see that $\mathrm{Hom}_R(A, U) = \mathrm{Hom}_R(A, E(R/M))$.

We need the following two lemmas for the proof of Theorem 2.8. Their proofs are routine and are therefore omitted.

Lemma 2.6. *Let $S = R_1 \oplus \cdots \oplus R_k$ where each R_i is a ring. Let $E = E_1 \oplus \cdots \oplus E_k$ be an S-module where each E_i is an injective R_i-module. Then E is an injective S-module.*

Lemma 2.7. *Let $S = R_1 \oplus \cdots \oplus R_k$ where each R_i is a local commutative Noetherian ring with maximal ideal M_i. Let $P_1, \cdots, P_k$ be the corresponding maximal ideals of S and let A be an Artinian S-module. Then $X_{P_i}(A)$ is an Artinian R_i-module for each i.*

Theorem 2.8. *Let R be a commutative Noetherian ring and let $M_1, \cdots, M_k$ be a fixed set of maximal ideals of R. Let $U = E(\bigoplus_{i=1}^k R/M_i)$ and $S = \mathrm{Hom}_R(U, U)$. Then there is a category equivalence between the category of Artinian R-modules belonging to $\{M_i\}$ and the category of Noetherian S-modules. The correspondence follows*:

(1) *If A is an Artinian R-module belonging to $\{M_i\}$ then $\mathrm{Hom}_R(A, U)$ is a Noetherian S-module and we have $A \cong \mathrm{Hom}_S(\mathrm{Hom}_R(A, U), U)$.*

(2) *If B is an S-module, then B is a Noetherian S-module if and only if $\mathrm{Hom}_S(B, U)$ is an Artinian R-module belonging to $\{M_i\}$. When this happens we have $B \cong \mathrm{Hom}_R(\mathrm{Hom}_S(B, U), U)$.*

Proof. Let A be an Artinian R-module belonging to $\{M_i\}$. Then we may write $A \cong A_{M_1} \oplus \cdots \oplus A_{M_k}$ and we have the isomorphism

$$(*) \qquad \mathrm{Hom}_R(A, U) \cong \bigoplus_{i=1}^{k} \mathrm{Hom}_{\hat{R}_{M_i}}(A_{M_i}, U_i)$$

where $U_i = E(R/M_i)$ [10, Proposition 4]. Since $S \cong \hat{R}_{M_1} \oplus \cdots \oplus \hat{R}_{M_k}$ it follows that $\mathrm{Hom}_R(A, U)$ is a Noetherian S-module. U is an injective S-module by Lemma 2.6, and it is easy to see that U is a minimal injective cogenerator for S. Now we apply the functor $\mathrm{Hom}_S(-, U)$ to $(*)$ and obtain isomorphisms

$$\begin{aligned}\mathrm{Hom}_S(\mathrm{Hom}_R(A, U), U) &\cong \bigoplus_{i=1}^{k} \mathrm{Hom}_S(\mathrm{Hom}_{\hat{R}_{M_i}}(A_{M_i}, U_i), U)\\ &\cong \bigoplus_{i=1}^{k} \mathrm{Hom}_S(\mathrm{Hom}_S(A_{M_i}, U), U) \cong \bigoplus_{i=1}^{k} A_{M_i} \cong A.\end{aligned}$$

The isomorphisms follow because $\mathrm{Hom}_{\hat{R}_{M_i}}(A_{M_i}, U_i) = \mathrm{Hom}_S(A_{M_i}, U)$ and by Proposition 2.5. This proves part (1).

Now let B be a Noetherian S-module. Then $\mathrm{Hom}_S(B, U)$ is an Artinian S-module by Proposition 2.5. For each i let P_i be the maximal ideal of S corresponding to M_i. It then follows from Lemma 2.7 that the P_i-primary component H_i, of $\mathrm{Hom}_S(B, U)$, is an Artinian $\hat{R}_{M_i}$-module. Therefore there exists an integer n such that $H_i \subset U_i^n$. Therefore H_i is an Artinian R-module because the R-structure and the $\hat{R}_{M_i}$-structure of U_i are the same. Hence $\mathrm{Hom}_S(B, U)$ is an Artinian R-module belonging to $\{M_i\}$. Now if C is any Artinian R-module belonging to $\{M_i\}$ then

$$C \otimes_R S \cong \left(\bigoplus_{i=1}^{k} C_{M_i}\right) \otimes_R \left(\bigoplus_{i=1}^{k} \hat{R}_{M_i}\right) \cong \bigoplus_{i=1}^{k} (C_{M_i} \otimes_R \hat{R}_{M_i}) \cong \bigoplus_{i=1}^{k} C_{M_i} \cong C.$$

Therefore $\mathrm{Hom}_S(B, U) \otimes_R S \cong \mathrm{Hom}_S(B, U)$. So by Proposition 2.5 we have

$$B \cong \mathrm{Hom}_S(\mathrm{Hom}_S(B, U), U) \cong \mathrm{Hom}_S(\mathrm{Hom}_S(B, U) \otimes_R S, U)$$
$$\cong \mathrm{Hom}_R(\mathrm{Hom}_S(B, U), \mathrm{Hom}_S(S, U)) \cong \mathrm{Hom}_R(\mathrm{Hom}_S(B, U), U).$$

Now suppose that B is an S-module such that $\mathrm{Hom}_S(B, U)$ is an Artinian R-module belonging to $\{M_i\}$. By looking at the M_i-primary components it is easy to see that $\mathrm{Hom}_S(B, U)$ is an Artinian S-module. So by Proposition 2.5 $\mathrm{Hom}_S(\mathrm{Hom}_S(B, U), U)$ is a Noetherian S-module. Since U is an S-cogenerator we have the exact sequence $0 \to B \to \mathrm{Hom}_S(\mathrm{Hom}_S(B, U), U)$. Therefore B is a Noetherian S-module. This proves part (2).

Remark. Let R be a commutative semilocal Noetherian ring and S the completion of R in the J-adic topology where J is the Jacobson radical of R. Then there is a category equivalence between the category of Artinian R-modules and the category of Noetherian S-modules as described in Theorem 2.8. Further, the converse of part (1) of Theorem 2.8 is also true in this case.

Corollary 2.9. *Let R be a commutative Noetherian ring and A an Artinian R-module. Then $\mathrm{Ext}_R^n(-, A)$ is convertible for all n.*

Proof. Let $M_1, \cdots, M_k$ be the maximal ideals of R belonging to A. Set $U = E(\bigoplus_{i=1}^k R/M_i)$ and $S = \mathrm{Hom}_R(U, U)$. Therefore by Theorem 2.8 we have $A \cong \mathrm{Hom}_S(\mathrm{Hom}_R(A, U), U)$. But U is an injective S-module by Lemma 2.6. The result now follows from Proposition 1.1.

Corollary 2.10. *Let R be a commutative semilocal Noetherian ring which is complete in the J-adic topology where J is the Jacobson radical of R. If A is a finitely generated R-module then $\mathrm{Ext}_R^n(-, A)$ is convertible for all n.*

Proof. It follows from Proposition 2.5 that A is U-reflexive where U is a minimal injective cogenerator for R.

Remark. Theorem 2.8 and Corollary 2.9 are both true under the more general hypothesis that R is a commutative ring such that R_M is a Noetherian ring for each maximal ideal M of R. If R is such a ring then $E(R/M)$ is an Artinian R-module for each maximal ideal M of R [17, Theorem 2]. So if A is an Artinian R-module then there exist maximal ideals $M_1, \cdots, M_k$ such that $A = X_{M_1}(A) \oplus \cdots \oplus X_{M_k}(A)$. Since each R_{M_i} is a Noetherian ring and $X_{M_i}(A)$ is an Artinian R_{M_i}-module the same proofs work.

Proposition 2.11. *Let R and S be rings and $\{B_\alpha\}$ a direct system of R-S bimodules with R acting on the left and S acting on the right. Let C be an injective right S-module and set $X_\alpha = \mathrm{Hom}_S(B_\alpha, C)$. Then $\mathrm{Ext}_R^n(A, \varprojlim X_\alpha) \cong \varprojlim \mathrm{Ext}_R^n(A, X_\alpha)$ for all R-modules A and all n.*

Proof. Since $\varprojlim \mathrm{Hom}_S(B_\alpha, C) \cong \mathrm{Hom}_S(\varinjlim B_\alpha, C)$ the proof is the same as the proof of Proposition 1.1.

Corollary 2.12. *Let R be a commutative ring, E an injective R-module and $\{A_\alpha\}$ an inverse system of R-modules each of which is E-reflexive. If $A = \varprojlim A_\alpha$ then $\mathrm{Ext}^n_R(-, A)$ is convertible for all n.*

Proof. Let $\{X_\beta\}$ be a direct system of R-modules. Then we have

$$\begin{aligned}\mathrm{Ext}^n_R(\varinjlim X_\beta, A) &= \mathrm{Ext}^n_R(\varinjlim X_\beta, \varprojlim A_\alpha) \cong \varprojlim \mathrm{Ext}^n_R(\varinjlim X_\beta, A_\alpha)\\ &\cong \varprojlim_\alpha \left(\varprojlim_\beta \mathrm{Ext}^n_R(X_\beta, A_\alpha)\right) \cong \varprojlim_\beta \left(\varprojlim_\alpha \mathrm{Ext}^n_R(X_\beta, A_\alpha)\right)\\ &\cong \varprojlim \mathrm{Ext}^n_R(X_\beta, \varprojlim A_\alpha)\\ &= \varprojlim \mathrm{Ext}^n_R(X_\beta, A).\end{aligned}$$

Definition. Let R be a ring and A an R-module. Then A is called *linearly compact* if there is a linear Hausdorff topology on A and if, with respect to this topology, any finitely solvable system of congruences $\{x \equiv x_\alpha \pmod{A_\alpha}\}$ is solvable, where the A_α are closed submodules of A. A is called *strictly linearly compact* if it is linearly topologized and has a fundamental system of neighborhoods of 0 consisting of submodules $\{A_\alpha\}$ such that each A/A_α is Artinian and A is complete in this topology. A is called *pseudocompact* if it is strictly linearly compact and in addition each A/A_α has finite length. We note that an Artinian module is linearly compact in the discrete topology [18, Proposition 5].

Remarks. (1) If R is a commutative ring and A is a pseudocompact R-module then $\mathrm{Ext}^n_R(-, A)$ is convertible for all n. For there is a fundamental system $\{A_\alpha\}$ of neighborhoods of 0 such that $A = \varprojlim A/A_\alpha$, where each A/A_α has finite length. But by Corollary 2.3 each A/A_α is U-reflexive where U is a minimal injective cogenerator. So the result follows from Corollary 2.12.

(2) Let R be a commutative Noetherian ring and A a strictly linearly compact R-module. Then there is a fundamental system $\{A_\alpha\}$ of neighborhoods of 0 such that $A = \varprojlim A/A_\alpha$, where each A/A_α is Artinian. If all the modules A/A_α belong to the same finite set of maximal ideals of R, then Proposition 2.11 combines with Theorem 2.8 to show that $\mathrm{Ext}^n_R(-, A)$ is convertible for all n.

Proposition 2.13. *Let R be a commutative ring with a cogenerator that is linearly compact in the discrete topology. Let A be an R-module that is linearly compact in the discrete topology. Then $\mathrm{Ext}^n_R(-, A)$ is convertible for all n.*

Proof. Let U be a cogenerator that is linearly compact in the discrete topology and set $S = \mathrm{Hom}_R(U, U)$. U is a right S-module in the usual way by writing the elements of S on the right. It then follows from [14, Corollary 1 of Theorem 2] that U is an injective right S-module. But by [14, Corollary 2 of Theorem 2] it follows that A is linearly compact in the discrete topology if and only if $A \cong \text{Hom}_S(\text{Hom}_R(A, U), U)$. The result now follows from Proposition 1.1.

Corollary 2.14. *Let R be a commutative semilocal Noetherian ring and A an R-module that is linearly compact in the discrete topology. Then* $\text{Ext}_R^n(-, A)$ *is convertible for all n.*

Proof. A minimal injective cogenerator for a commutative semilocal Noetherian ring is Artinian and thus linearly compact in the discrete topology.

3. **Characterizations.** We begin with a proposition that produces many examples to show that Ext is not convertible.

Proposition 3.1. *Let R be a ring with a nonprojective flat R-module. Then there exists an R-module A such that* $\text{Ext}_R^1(-, A)$ *is not convertible.*

Proof. Let X be a nonprojective flat R-module. Since X is flat it can be written $X = \varinjlim X_\alpha$ where $\{X_\alpha\}$ is a direct system of finitely generated free R-modules [7]. Since X is not projective there exists an R-module A such that $\text{Ext}_R^1(X, A) \neq 0$. But $\text{Ext}_R^1(X_\alpha, A) = 0$ for each X_α. Therefore $\varprojlim \text{Ext}_R^1(X_\alpha, A) = 0$ and $\text{Ext}_R^1(\varinjlim X_\alpha, A) \neq 0$.

Remarks. (1) If R is an integral domain such that $\text{Ext}_R^1(-, A)$ is convertible for all R-modules A then R is a field. For if R were not equal to its quotient field Q then we would obtain a contradiction to Proposition 3.1 because Q would be a nonprojective flat R-module.

(2) If R is a commutative ring of finite global dimension such that $\text{Ext}_R^1(-, A)$ is convertible for all R-modules A then R is a semisimple Artinian ring. The convertibility assumption implies that every flat R-module is projective. Since R is commutative it follows that every module has projective dimension 0 or ∞ [1]. Hence every module is projective so that R is semisimple Artinian.

(3) Since Ext_R^1 vanishes when R is semisimple Artinian the converses of Remarks (1) and (2) are trivially true. It seems reasonable to conjecture that if R is a ring such that $\text{Ext}_R^1(-, A)$ is convertible for all R-modules A then R must be semisimple Artinian.

(4) We also note here that if R is a ring and n is a positive integer such that $\text{Ext}_R^n(-, A)$ is convertible for all R-modules A that are an image of an injective, then $\text{Ext}_R^k(-, B)$ is convertible for all R-modules B and all $k > n$. This follows from the exact sequence $0 \to B \to E(B) \to E(B)/B \to 0$. For then we obtain the isomorphisms

$$\operatorname{Ext}_R^{n+1}(\varinjlim X_\alpha, B) \cong \operatorname{Ext}_R^n(\varinjlim X_\alpha, E(B)/B) \cong \varprojlim \operatorname{Ext}_R^n(X_\alpha, E(B)/B)$$
$$\cong \varprojlim \operatorname{Ext}_R^{n+1}(X_\alpha, B).$$

For the next result we need a lemma.

Lemma 3.2. *Let R be a commutative ring and I a nonzero finitely generated ideal contained in the Jacobson radical of R. If A is an Artinian R-module then* $A \cong \varinjlim \operatorname{Hom}_R(R/I^n, A)$.

Proof. Since $\operatorname{Hom}_R(R/I^n, A) \cong \operatorname{Ann}_A(I^n)$ we need only show that $A = \bigcup_{n=1}^\infty \operatorname{Ann}_A(I^n)$. Let $x \in A$. For each $n > 0$ the submodule $I^n x \subset A$ is finitely generated. Since A is Artinian the descending chain $Ix \supset I^2x \supset \cdots \supset I^n x \supset \cdots$ must stop. So there exists $k > 0$ such that $I^k x = I^{k+1}x = I(I^k x)$. Therefore $I^k x = 0$ by the Nakayama lemma. Hence $x \in \operatorname{Ann}_A(I^k)$. Thus

$$A = \bigcup_{n=1}^{\infty} \operatorname{Ann}_A(I^n) \cong \varinjlim \operatorname{Ann}_A(I^n) \cong \varinjlim \operatorname{Hom}_R(R/I^n, A).$$

Remark. If I is a finitely generated ideal of R contained in the Jacobson radical and B and C are R-modules such that $\operatorname{Hom}_R(B, C)$ is Artinian, then $\operatorname{Hom}_R(B, C) \cong \varinjlim \operatorname{Hom}_R(B/I^nB, C)$.

Proposition 3.3. *Let R be a commutative semilocal Noetherian ring, J the Jacobson radical of R, U a minimal injective cogenerator and A a finitely generated R-module. Then there is a natural isomorphism* σ: $\operatorname{Hom}_R(\operatorname{Hom}_R(A, U), U) \to \varprojlim A/J^nA$ *such that the following diagram is commutative:*

$$\begin{array}{ccc} A & \xrightarrow{\phi} & \operatorname{Hom}_R(\operatorname{Hom}_R(A, U), U) \\ & {\scriptstyle\rho}\searrow & \downarrow{\scriptstyle\sigma} \\ & & \varprojlim A/J^nA \end{array}$$

where ϕ and ρ are the natural maps defined by $\phi(a)(f) = f(a)$ and $\rho(a) = (a + J^nA)$ for all $a \in A$ and $f \in$ $\operatorname{Hom}_R(A, U)$.

Proof. Since U is Artinian and A is finitely generated it follows that $\operatorname{Hom}_R(A, U)$ is Artinian. Therefore there is an isomorphism α: $\varinjlim \operatorname{Hom}_R(A/J^nA, U) \to \operatorname{Hom}_R(A, U)$. To describe α we first recall for each k the isomorphisms described below:

$$\begin{array}{ccccc} \operatorname{Hom}_R(A/J^kA, U) & \cong & \operatorname{Hom}_R(R/J^k, \operatorname{Hom}_R(A, U)) & \cong & \operatorname{Ann}_{A^*}(J^k) \subset \operatorname{Hom}_R(A, U) \\ f_k & \leftrightarrow & h_k & \leftrightarrow & h_k(1 + J^k) \end{array}$$

where $h_k(r + J^k)(a) = f_k(ra + J^kA)$ for $r \in R$ and $a \in A$. Let S denote the relations in the direct limit and recall that any element in $\varinjlim \mathrm{Hom}_R(A/J^nA, U)$ has the form $f_k + S$ where $f_k \in \mathrm{Hom}_R(A/J^kA, U)$ for some integer k. Then $\alpha(f_k + S) = h_k(1 + J^k)$. Now apply the functor $\mathrm{Hom}_R(-, U)$ to the isomorphism α to obtain the isomorphism

$$\alpha^*: \mathrm{Hom}_R(\mathrm{Hom}_R(A, U), U) \longrightarrow \mathrm{Hom}_R(\varinjlim \mathrm{Hom}_R(A/J^nA, U), U)$$

where as usual $\alpha^*(f) = f \circ \alpha$ for all $f \in \mathrm{Hom}_R(\mathrm{Hom}_R(A, U), U)$. Since $\mathrm{Hom}_R(-, U)$ is convertible we have the isomorphism

$$\beta: \mathrm{Hom}_R(\varinjlim \mathrm{Hom}_R(A/J^nA, U), U) \longrightarrow \varprojlim \mathrm{Hom}_R(\mathrm{Hom}_R(A/J^nA, U), U)$$

given by $\beta(g) = (g_n)$ where $g \in \mathrm{Hom}_R(\varinjlim \mathrm{Hom}_R(A/J^nA, U), U)$ and $g(f_k + S) = g_k(f_k)$ for all $f_k + S \in \varinjlim \mathrm{Hom}_R(A/J^nA, U)$ and $g_k \in \mathrm{Hom}_R(\mathrm{Hom}_R(A/J^kA, U), U)$. Since A is a finitely generated R-module it follows that A/J^nA has finite length for all $n > 0$. Therefore each A/J^nA is U-reflexive by Corollary 2.3. Hence we have an isomorphism

$$\gamma: \varprojlim \mathrm{Hom}_R(\mathrm{Hom}_R(A/J^nA, U), U) \longrightarrow \varprojlim A/J^nA$$

given by $\gamma((g_n)) = (a_n + J^nA)$ where $g_n = \phi_n(a_n + J^nA)$ and ϕ_n is the natural isomorphism $\phi_n: A/J^nA \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(A/J^nA, U), U)$. Finally let $\sigma = \gamma \circ \beta \circ \alpha^*$. Then σ is an isomorphism because each of γ, β and α^* are isomorphisms. Let $F = \sigma \circ \phi$. We must show that $F = \rho$. Let $a \in A$. Then $F(a) = (\sigma \circ \phi)(a) = (\gamma \circ \beta \circ \alpha^* \circ \phi)(a) = \gamma \circ \beta \circ \alpha^*(\phi(a)) = \gamma \circ \beta(\phi(a) \circ \alpha) = \gamma((g_n))$ where $(\phi(a) \circ \alpha)(f_k + S) = g_k(f_k)$ for all $f_k + S \in \varinjlim \mathrm{Hom}_R(A/J^nA, U)$. But $(\phi(a) \circ \alpha)(f_k + S) = \phi(a)(\alpha(f_k + S)) = \phi(a)(h_k(1 + J^k)) = h_k(1 + J^k)(a) = f_k(a + J^kA) = \phi_k(a + J^kA)(f_k)$. Therefore $g_k = \phi_k(a + J^kA)$ for all k. Hence $F(a) = \gamma((\phi_n(a + J^kA))) = (a_n + J^nA)$ where $\phi_n(a_n + J^nA) = \phi_n(a + J^nA)$ for all n. But each ϕ_n is an isomorphism. Therefore $a + J^nA = a_n + J^nA$ for all n. Thus $F(a) = (a + J^nA) = \rho(a)$. Therefore $F = \rho$ and the proof is finished.

For the next result we need a definition. A ring R is called *coherent* if every direct product of flat R-modules is a flat R-module. Noetherian rings as well as semihereditary rings are coherent [4]. The idea for the following proposition comes from [6, Theorem 8.1].

Proposition 3.4. *Let R be a commutative coherent ring, I a finitely generated ideal of R and A an R-module such that $\mathrm{Ext}_R^1(-, A)$ is convertible. Then the following sequence is exact:*

$$0 \longrightarrow \bigcap I^nA \longrightarrow A \xrightarrow{\rho} \varprojlim A/I^nA \longrightarrow 0$$

where ρ is the natural map.

Proof. Since $\mathrm{Ext}_R^1(-, A)$ is convertible it follows that $\mathrm{Ext}_R^1(F, A) = 0$ for all flat R-modules F. Throughout this proof we will use the following notation: If B is an R-module then $\Pi B = \Pi_{i=0}^{\infty} B_i$ and $\bigoplus B = \bigoplus_{i=0}^{\infty} B_i$ where $B_i = B$ for each integer $i \geq 0$. Since R is coherent it follows that ΠR is a flat R-module. For each integer $n \geq 0$ set $S_n = \Pi R$ and whenever $n \leq m$ we define $f_{n,m}\colon S_n \to S_m$ by the following: For each $(r_0, r_1, \cdots) \in S_n$ let $f_{n,m}((r_0, r_1, \cdots)) = (0, \cdots, 0, r_m, r_{m+1}, \cdots)$. Then $\{S_n, f_{n,m}\}$ is a direct system of R-modules whose direct limit is isomorphic to $\Pi R/\bigoplus R$. Since each S_n is flat and a direct limit of flat modules is flat it follows that $\Pi R/\bigoplus R$ is a flat R-module. Therefore $\mathrm{Ext}_R^1(\Pi R/\bigoplus R, A) = 0$. Hence we have the following exact sequence:

$$0 \to \mathrm{Hom}_R(\Pi R/\bigoplus R, A) \to \mathrm{Hom}_R(\Pi R, A) \to \mathrm{Hom}_R(\bigoplus R, A) \to 0.$$

Since $\mathrm{Hom}_R(-, A)$ is convertible we have the exact sequence

$$\mathrm{Hom}_R(\Pi R, A) \xrightarrow{\alpha} \Pi A \to 0$$

where for each $(a_n) \in \Pi A$ there exists $g \in \mathrm{Hom}_R(\Pi R, A)$ such that $\alpha(g) = (a_n)$ and $g(e_n) = a_n$ for all $n \geq 0$ where e_n is the element in ΠR all of whose components are zero except a 1 in the nth place. Let $I = (x_1, \cdots, x_k)$ be an ideal of R with generators $x_1, \cdots, x_k$. For each $n \geq 1$ set $I_n = (x_1^n, \cdots, x_k^n)$. It is clear that $I_n \subset I^n$ and it is easy to see that $I^{(n-1)k+1} \subset I_n$. Therefore $\bigcap I^n A = \bigcap I_n A$ and $\varprojlim A/I^n A = \varprojlim A/I_n A$. So it is sufficient to show that the following sequence is exact:

$$0 \to \bigcap I_n A \to A \xrightarrow{\rho} \varprojlim A/I_n A \to 0.$$

Let $a \in \varprojlim A/I_n A$. Then $a = (a_0 + IA, a_1 + I_2 A, \cdots) = (a_n + I_{n+1}A)$. It is easy to see that for each $n > 0$ we may write $a_n = a_0 + \Sigma_{j=1}^n (\Sigma_{i=1}^k x_i^j a_{ji})$ where $a_{ji} \in A$. Now let $b = (a_0, a_{11}, a_{12}, \cdots, a_{1k}, a_{21}, a_{22}, \cdots, a_{2k}, \cdots) \in \Pi A$. Then there exists $g \in \mathrm{Hom}_R(\Pi R, A)$ such that $g(e_0) = a_0$, $g(e_1) = a_{11}$, $g(e_2) = a_{12}, \cdots$, and in general $g(e_{(j-1)k+i}) = a_{ji}$. Now let $d = (1, x_1, x_2, \cdots, x_k, x_1^2, x_2^2, \cdots, x_k^2, \cdots) \in \Pi R$. We may then write $d = e_0 + \Sigma_{i=1}^k x_i s_i$ where each $s_i \in \Pi R$, and for each $n \geq 1$ we have $d = e_0 + \Sigma_{j=1}^n (\Sigma_{i=1}^k x_i^j e_{(j-1)k+i}) + \Sigma_{i=1}^k x_i^{n+1} t_i$ where each $t_i \in \Pi R$. Then $g(d) = g(e_0) + \Sigma_{i=1}^k x_i g(s_i) \equiv a_0 \pmod{IA}$ and for each $n \geq 1$ we have

$$\begin{aligned} g(d) &= g(e_0) + \sum_{j=1}^n \left(\sum_{i=1}^k x_i^j g(e_{(j-1)k+i}) \right) + \sum_{i=1}^k x_i^{n+1} g(t_i) \\ &= a_0 + \sum_{j=1}^n \left(\sum_{i=1}^k x_i^j a_{ji} \right) + \sum_{i=1}^k x_i^{n+1} g(t_i) \equiv a_n \pmod{I_{n+1}A}. \end{aligned}$$

Therefore $\rho(g(d)) = (g(d) + I_{n+1}A) = (a_n + I_{n+1}A) = a$. So the natural map $\rho: A \to \varprojlim A/I_nA$ is surjective. But $\mathrm{Ker}\,\rho = \bigcap I_nA$ which gives the desired exact sequence.

The next proposition shows that if $\mathrm{Ext}_R^1(-, A)$ is convertible then it is a "completion" functor in some cases. This property will also be demonstrated in later results.

Proposition 3.5. *Let R be a commutative semilocal Noetherian ring and A a finitely generated R-module. The following statements are equivalent:*

(a) *A is complete in the J-adic topology where J is the Jacobson radical of R.*

(b) *A is U-reflexive where U is a minimal injective cogenerator.*

(c) *A is linearly compact in the discrete topology.*

(d) $\mathrm{Ext}_R^n(-, A)$ *is convertible for all n.*

(e) $\mathrm{Ext}_R^1(-, A)$ *is convertible.*

Proof. (a) $\Rightarrow$ (b) This follows from Proposition 3.3 since ρ is an isomorphism if and only if ϕ is an isomorphism.

(b) $\Rightarrow$ (c) Let $S = \mathrm{Hom}_R(U, U)$ and let $g \in \mathrm{Hom}_S(\mathrm{Hom}_R(A, U), U)$. Since R is contained in S and g is an S-homomorphism it follows that g is an R-homomorphism. Since A is U-reflexive there exists an element $a \in A$ such that $g = \phi(a)$ where $\phi: A \to \mathrm{Hom}_R(\mathrm{Hom}_R(A, U), U)$ is the natural isomorphism. Therefore $A \cong \mathrm{Hom}_S(\mathrm{Hom}_R(A, U), U)$ via ϕ. Hence A is linearly compact in the discrete topology by [14, Corollary 2 of Theorem 2].

(c) $\Rightarrow$ (d) This follows from Corollary 2.14.

(d) $\Rightarrow$ (e) Trivial.

(e) $\Rightarrow$ (a) This follows from Proposition 3.4.

Remark. In the situation of Proposition 3.5 let $0 \to A \to B \to C \to 0$ be an exact sequence of finitely generated R-modules. Then $\mathrm{Ext}_R^1(-, B)$ is convertible if and only if $\mathrm{Ext}_R^1(-, A)$ and $\mathrm{Ext}_R^1(-, C)$ are both convertible.

Definition. A ring R has a *Morita-duality* if there exists a ring S and an S-R bimodule U such that U is an injective cogenerator as a left S-module and as a right R-module, and $R = \mathrm{Hom}_S(U, U)$ and $S = \mathrm{Hom}_R(U, U)$.

Remarks. (1) As a consequence of the definition we see that if R is a ring with a Morita-duality then $\mathrm{Hom}(-, U)$ establishes a category equivalence between the category of U-reflexive right R-modules and the category of U-reflexive left S-modules. It is also clear that the finitely generated modules are U-reflexive.

(2) It follows from [13, Theorem 2] that if R is a ring with a Morita-duality induced by the injective cogenerator U, then the U-reflexive modules are exactly the modules that are linearly compact in the discrete topology. Therefore all submodules of a finitely generated module are linearly compact in the discrete topology.

Proposition 3.6. *Let R be a commutative ring with a Morita-duality and let A be an R-module that is linearly compact in the discrete topology. Then $\mathrm{Ext}_R^n(-, A)$ is convertible for all n.*

Proof. Since R is commutative it has a Morita-duality with itself [13, Theorem 3]. This means that there exists an injective cogenerator U such that $R = \mathrm{Hom}_R(U, U)$. Since A is linearly compact in the discrete topology it is U-reflexive. The result now follows from Proposition 2.1.

Lemma 3.7. *Let R, S and T be rings such that $R = S \oplus T$ and suppose that $\mathrm{Ext}_R^n(-, R)$ is convertible. Then $\mathrm{Ext}_S^n(-, S)$ and $\mathrm{Ext}_T^n(-, T)$ are both convertible.*

Proof. Let A be an S-module. Then A is an R-module via the projection map $R \to S$. Since $\mathrm{Hom}_R(S, S) = \mathrm{Hom}_S(S, S) \cong S$ it follows from [3, Chapter VI, Proposition 4.1.4] that $\mathrm{Ext}_S^n(A, S) \cong \mathrm{Ext}_R^n(A, S)$. Since T is contained in $\mathrm{Ann}_R(A)$ it follows that $\mathrm{Ext}_R^n(A, T) = 0$. Therefore we have $\mathrm{Ext}_R^n(A, R) \cong \mathrm{Ext}_R^n(A, S) \oplus \mathrm{Ext}_R^n(A, T) \cong \mathrm{Ext}_S^n(A, S)$. It is now clear that $\mathrm{Ext}_S^n(-, S)$ is convertible, and the same argument shows that $\mathrm{Ext}_T^n(-, T)$ is convertible.

Theorem 3.8. *Let R be a commutative Noetherian ring. The following statements are equivalent:*

(a) *R is semilocal and complete in the J-adic topology where J is the Jacobson radical of R.*

(b) *R has a Morita-duality.*

(c) *There exists an injective R-module C such that R is C-reflexive.*

(d) *$\mathrm{Ext}_R^n(-, R)$ is convertible for all n.*

(e) *$\mathrm{Ext}_R^1(-, R)$ is convertible.*

Proof. (a) $\Rightarrow$ (b) Let U be a minimal injective cogenerator for R. Since R is complete in the J-adic topology it follows by Proposition 3.3 that R is U-reflexive. Therefore R has a Morita-duality.

(b) $\Rightarrow$ (c) Since R has a Morita-duality it has one with itself. So there exists an injective cogenerator C such that R is C-reflexive.

(c) $\Rightarrow$ (d) This follows from Proposition 2.1.

(d) $\Rightarrow$ (e) Trivial.

(e) $\Rightarrow$ (a) Let M be a maximal ideal of R. Since $\mathrm{Ext}_R^1(-, R)$ is convertible it follows from Proposition 3.4 that the sequence $0 \to \bigcap M^n \to R \to \varprojlim R/M^n \to 0$ is exact. Set $\hat{R}_0 = \varprojlim R/M^n$. Then $\hat{R}_0$ is a complete local ring and a cyclic R-module. Since completion is flat it follows that $\hat{R}_0$ is a finitely generated flat R-module and is therefore a projective R-module. Hence there exists a ring R_1 such that $R \cong \hat{R}_0 \oplus R_1$. If $R_1 = 0$ we are done. If $R_1 \neq 0$ then $\mathrm{Ext}_{R_1}^1(-, R_1)$

is convertible by Lemma 3.7. So we choose a maximal ideal M_1 of R_1 and repeat the above procedure to find a ring R_2 such that $R \cong \hat{R}_0 \oplus \hat{R}_1 \oplus R_2$ where $\hat{R}_1$ is a complete local ring. If $R_2 = 0$ we are done. If $R_2 \neq 0$ we do the same thing as before. Since R is Noetherian the procedure must stop so that there exists an integer $n \geq 0$ such that $R \cong \hat{R}_0 \oplus \hat{R}_1 \oplus \cdots \oplus \hat{R}_n$ where each $\hat{R}_i$ is a complete local ring. But a finite direct sum of complete local rings is semilocal and complete in the J-adic topology where J is the Jacobson radical of R.

Remark. In the situation of Theorem 3.8 consider the statement (f): There exists an injective R-module C such that every cyclic R-module is C-reflexive. It is clear that (f) is equivalent to the other statements. The statement (f) $\Rightarrow$ (a) is a remark of Matlis [8, Remark 2 following Theorem 4.2]. So we see that the converse is true.

Notation. Let R be an integral domain with quotient field Q. We denote by K the R-module Q/R. Then the following sequence is exact:

$$(*) \qquad 0 \to R \xrightarrow{i} \mathrm{Hom}_R(K, K) \to \mathrm{Ext}_R^1(Q, R) \to 0$$

where i is a ring homomorphism defined by $i(r)(x) = rx$ for all $r \in R$ and $x \in K$ [11, Proposition 5.2].

Proposition 3.9. *If R is an integral domain with a Morita-duality then there is a ring isomorphism $R \cong \mathrm{Hom}_R(K, K)$ and every element of $\mathrm{Hom}_R(K, K)$ is given by multiplication of an element of R.*

Proof. Since R has a Morita-duality there exists an injective cogenerator U such that $R = \mathrm{Hom}_R(U, U)$. Therefore $\mathrm{Ext}_R^1(-, R)$ is convertible which yields $\mathrm{Ext}_R^1(Q, R) = 0$ since Q is a flat R-module. So the result follows from exact sequence $(*)$.

Definition. An integral domain R is called *reflexive* if every submodule of a finitely generated torsion-free R-module is R-reflexive. R is called *completely reflexive* if every reduced (no nonzero divisible submodules) torsion-free R-module of finite rank is R-reflexive. Matlis showed that R is reflexive if and only if K is a minimal injective cogenerator [12, Theorem 2.1], and that a reflexive domain R is completely reflexive if and only if $R \cong \mathrm{Hom}_R(K, K)$ [12, Proposition 5.1]. It is clear that a completely reflexive domain is reflexive. A Dedekind ring is reflexive. The ring of formal power series in one variable over a field is completely reflexive. More generally, any complete discrete valuation ring is completely reflexive.

Proposition 3.10. *Let R be a reflexive domain. The following statements are equivalent:*

(a) *R is completely reflexive.*

(b) *R has a Morita-duality.*

(c) *There exists an injective R-module C such that R is C-reflexive.*

(d) $\mathrm{Ext}_R^n(-, R)$ *is convertible for all n.*

(e) $\mathrm{Ext}_R^1(-, R)$ *is convertible.*

Proof. (a) ⟹ (b) $R \cong \mathrm{Hom}_R(K, K)$ where K is a minimal injective cogenerator.

(b) ⟹ (c) Let C be the injective cogenerator that gives R a Morita-duality.

(c) ⟹ (d) This follows from Proposition 2.1.

(d) ⟹ (e) Trivial.

(e) ⟹ (a) This follows from exact sequence (*).

Definition. A valuation ring R is called *almost maximal* if every proper homomorphic image of Q is linearly compact in the discrete topology, while R is *maximal* if Q is linearly compact in the discrete topology. Matlis showed that an almost maximal valuation ring R is maximal if and only if $R \cong \mathrm{Hom}_R(K, K)$ if and only if $R \cong \mathrm{Hom}_R(U, U)$ where U is a minimal injective cogenerator [9, Lemma 7 and Theorem 9]. So the proof of the next proposition is the same as the proof of Proposition 3.10.

Proposition 3.11. *Let R be an almost maximal valuation ring. The following statements are equivalent:*

(a) *R is maximal.*

(b) *R has a Morita-duality.*

(c) *There exists an injective R-module C such that R is C-reflexive.*

(d) $\mathrm{Ext}_R^n(-, R)$ *is convertible for all n.*

(e) $\mathrm{Ext}_R^1(-, R)$ *is convertible.*

4. Particular rings and modules.

Proposition 4.1. *Let R be a semihereditary ring and A an R-module such that* $\mathrm{Ext}_R^n(-, A)$ *is convertible for some positive integer n. Then the injective dimension of A is* $\leq n$.

Proof. Let I be an ideal of R. We must show that $\mathrm{Ext}_R^{n+1}(R/I, A) = 0$. Since $\mathrm{Ext}_R^{n+1}(R/I, A) \cong \mathrm{Ext}_R^n(I, A)$ it is sufficient to show that $\mathrm{Ext}_R^n(I, A) = 0$. We may write $I = \varinjlim I_\alpha$ where $\{I_\alpha\}$ is the direct system of finitely generated ideals contained in I. Each I_α is a projective R-module since R is semihereditary. Therefore we have $\mathrm{Ext}_R^n(I, A) = \mathrm{Ext}_R^n(\varinjlim I_\alpha, A) \cong \varprojlim \mathrm{Ext}_R^n(I_\alpha, A) = 0$.

Corollary 4.2. *Let R be a commutative semihereditary ring (for example a Prüfer ring) and A an R-module of finite length. Then* $\mathrm{inj\,dim}_R\, A \leq 1$.

Proof. Corollary 2.4 and Proposition 4.1.

Proposition 4.3. *Let R be a Prüfer ring and A an R-module whose torsion submodule $t(A)$ has finite length. Then $t(A)$ is a direct summand of A.*

Proof. Let $\{X_\alpha\}$ be the direct system of finitely generated submodules of the torsion-free R-module $A/t(A)$. Each X_α is projective since R is a Prüfer ring. But $\operatorname{Ext}^1_R(-, t(A))$ is convertible by Corollary 2.4. Therefore

$$\operatorname{Ext}^1_R(A/t(A), t(A)) = \operatorname{Ext}^1_R(\varinjlim X_\alpha, t(A)) \cong \varprojlim \operatorname{Ext}^1_R(X_\alpha, t(A)) = 0.$$

Proposition 4.4. *Let R be a Dedekind ring and A an R-module whose torsion submodule $t(A)$ is Artinian. Then $t(A)$ is a direct summand of A.*

Proof. $\operatorname{Ext}^1_R(-, t(A))$ is convertible by Corollary 2.9 so the result follows just as in the proof of Proposition 4.3.

Proposition 4.5. *Let R be a commutative ring, U an injective R-module and $\{X_\alpha\}$ an inverse system of R-modules each of which is U-reflexive. Then* $\operatorname{inj\,dim}_R(\varprojlim X_\alpha) \leq \sup_\alpha\{\operatorname{inj\,dim}_R X_\alpha\}$.

Proof. This follows from Proposition 2.11.

Remarks. (1) If R is a commutative ring with a Morita-duality and $\{X_\alpha\}$ is an inverse system of R-modules each of which is linearly compact in the discrete topology, then $\operatorname{inj\,dim}_R(\varprojlim X_\alpha) \leq \sup_\alpha\{\operatorname{inj\,dim}_R X_\alpha\}$.

(2) If R is a Prüfer ring and $\{X_\alpha\}$ is an inverse system of R-modules each having finite length, then $\operatorname{inj\,dim}_R(\varprojlim X_\alpha) \leq 1$.

Proposition 4.6. *Let R be a commutative Noetherian ring, A an Artinian R-module and X any R-module. Then $\operatorname{Ext}^n_R(X, A)$ is a strictly linearly compact R-module for all n.*

Proof. Let $\{X_\alpha\}$ be the direct system of all finitely generated submodules of X. Then each of the R-modules $\operatorname{Ext}^n_R(X_\alpha, A)$ is Artinian and therefore strictly linearly compact. By Corollary 2.9 we have $\operatorname{Ext}^n_R(X, A) \cong \varprojlim \operatorname{Ext}^n_R(X_\alpha, A)$. The result now follows because an inverse limit of strictly linearly compact modules is strictly linearly compact [2, p. 111, Exercise 19c].

The next proposition offers an example of particular modules that provide counterexamples to the theory for Ext^2.

Proposition 4.7. *Let F be an uncountable field, X and Y indeterminates over F and $R = F[X, Y]_{(X,Y)}$, the localization of the ring $F[X, Y]$ at the maximal ideal (X, Y). Let $H = \operatorname{Hom}_R(K, K)$. Then*

(a) $\mathrm{Ext}_R^2(-, H)$ *is not convertible.*

(b) $\mathrm{Ext}_R^2(Q, -)$ *does not commute with all inverse limits.*

Proof. Gruson has shown that $\mathrm{Ext}_R^2(Q, R) \neq 0$ [5]. Therefore we also know that $\mathrm{Ext}_R^2(-, R)$ is not convertible. Now for any integral domain the functor $\mathrm{Ext}_R(Q, -)$ applied to exact sequence (*) $0 \to R \to H \to \mathrm{Ext}_R^1(Q, R) \to 0$ yields $\mathrm{Ext}_R^2(Q, R) \cong \mathrm{Ext}_R^2(Q, H)$. Therefore $\mathrm{Ext}_R^2(Q, H) \neq 0$ so that $\mathrm{Ext}_R^2(-, H)$ is not convertible. For any integral domain R, Matlis has shown that H is isomorphic to the completion of R in the R-topology [11, Proposition 6.4]. The R-topology on R has as a subbase for the neighborhoods of 0, the set of ideals $\{rR\}$ where $r \in R$, $r \neq 0$. Therefore $H \cong \varprojlim R/rR$. Since each R/rR is torsion of bounded order we have $\mathrm{Ext}_R^2(Q, R/rR) = 0$. Therefore $\varprojlim \mathrm{Ext}_R^2(Q, R/rR) = 0$ but $\mathrm{Ext}_R^2(Q, \varprojlim R/rR) \neq 0$.

Remark. We do not know of sufficient conditions on R and an R-module A such that $\mathrm{Ext}_R^n(A, -)$ commutes with all inverse limits of R-modules.

Finally we consider the case where there may be a restriction on both the direct system $\{X_\alpha\}$ and the module A.

Notation. Denote the pth right derived functor of $\varprojlim$ by $\varprojlim^{(p)}$. Let R be a ring, A an R-module and $\{X_\alpha\}$ a direct system of R-modules. We consider the following spectral sequence of Roos [15]:

$$E_2^{p,q} = \lim^{(p)} \underleftarrow{\mathrm{Ext}}_R^q(X_\alpha, A) \underset{p}{\Rightarrow} \mathrm{Ext}_R^n(\varinjlim X_\alpha, A).$$

A proof of the existence of this spectral sequence is given in [6, Theorem 4.2]. Using standard spectral sequence arguments [3, Chapter XV] we have the following proposition.

Proposition 4.8. *Let R be a ring, A an R-module and $\{X_\alpha\}$ a direct system of R-modules. For each integer q let $\varprojlim^{(p)} \mathrm{Ext}_R^q(X_\alpha, A) = 0$ for all $p \geq 2$. Then for each $n > 0$ the following sequence is exact:*

$$(**) \quad 0 \to \varprojlim^{(1)} \mathrm{Ext}_R^{n-1}(X_\alpha, A) \to \mathrm{Ext}_R^n(\varinjlim X_\alpha, A) \to \varprojlim \mathrm{Ext}_R^n(X_\alpha, A) \to 0.$$

Remarks. (1) Jensen has shown that $\varprojlim^{(p)} C_\alpha = 0$ for all $p \geq 2$ and all inverse systems $\{C_\alpha\}_{\alpha \in D}$ of R-modules when D is a countable directed set [6, Theorem 2.2]. Therefore (**) always holds when $\{X_\alpha\}$ is a direct system of R-modules and the index set is countable.

(2) If R is an integral domain and $\{X_\alpha\}$ is a direct system of R-modules over a countable directed set, then (**) holds and when $n = 1$ we have an isomorphism $\mathrm{Ext}_R^1(\varinjlim X_\alpha, A) \cong \varprojlim \mathrm{Ext}_R^1(X_\alpha, A)$ in the following two cases:

(a) $\{X_\alpha\}$ torsion and A torsion-free.

(b) $\{X_\alpha\}$ divisible and A reduced.

For in either case we have $\mathrm{Hom}_R(X_\alpha, A) = 0$.

(3) If R is a commutative hereditary ring, $\{X_\alpha\}$ a direct system of finitely generated R-modules and A an Artinian R-module, then $\mathrm{Ext}_R^1(\varinjlim X_\alpha, A) \cong \varprojlim \mathrm{Ext}_R^1(X_\alpha, A)$. For by using standard arguments we obtain the exact sequence (**) where $n = 1$. But each $\mathrm{Hom}_R(X_\alpha, A)$ is an Artinian R-module. Therefore $\varprojlim^{(p)} \mathrm{Hom}_R(X_\alpha, A) = 0$ for all $p > 0$ by [6, Corollary 7.2].

(4) Jensen [6] has general results on the vanishing of $\varprojlim^{(p)} C_\alpha$ for certain inverse systems $\{C_\alpha\}$ and all $p \geq 2$. So if $\{X_\alpha\}$ is a direct system and A is a module such that $\{\mathrm{Ext}_R^n(X_\alpha, A)\}$ has the same property as the $\{C_\alpha\}$ for all $n > 0$, then (**) holds.

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